TOPOLOGY - III, EXERCISE SHEET 14

Recall from the lectures the **Lefschetz fixed point theorem**: Let X be a finite $CW-$complex and let $f : X \to X$ be a continuous mapping, then one can define the Lefschetz number of f as the following alternating sum:

$$\tau(f) := \sum_i (-1)^i Tr\,(H_i(f, \mathbb{Q})).$$

Where $H_i(f, \mathbb{Q})$ is the group homomorphism $f_* : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$ induced on homology groups by f. Lefschetz fixed point theorem then states that if $\tau(f) \neq 0$ then the mapping f has fixed points.

A version of the Lefschetz fixed point theorem is the so called **Lefschetz-Hopf fixed point theorem** which says that if X^f, the fixed point set of f is a finite set of points then:

$$\tau(f) = \sum_{p \in X^f} \text{index}_x(f)$$

where, the index of a fixed point is a suitable count of the point with multiplicity. For example, it turns out that if f is a holomorphic function then $\text{index}_x(f)$ is the order of the holomorphic function $f(z) - x$ around x.

This exercise sheet details some applications of these fixed point theorems.

Exercise 1. Deduce the Brouwer fixed point theorem from the Lefschetz fixed point theorem. That is, show that every continuous mapping from D^n to itself has a fixed point.

Exercise 2. The finite CW-complex assumption is quite important for the Lefschetz fixed point theorem to work. Note that this assumption implies in particular that X is compact. Give an example of a non-compact space X and continuous map $f : X \to X$ with no fixed points but with $\tau(f) \neq 0$.

Exercise 3. Let $f : S^n \to S^n$ be a continuous map such that the degree of f is not equal to the degree of the anti-podal map. Show that f has a fixed point.

Exercise 4.

(1) Show that if n is even then every continuous mapping $f : \mathbb{RP}^n \to \mathbb{RP}^n$ has fixed points.
(2) Construct a continuous mapping $f : \mathbb{RP}^n \to \mathbb{RP}^n$ with a fixed point when n is odd.
Hint: Can you construct a linear map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ without a real eigenvalue.

Exercise 5. Given a topological space X, one can associate to X it's cohomology groups $H^i(X, A)$ with coefficients in a fixed Abelian group A. Similar to homology, given a continuous map $f : X \to Y$, there are induced group homomorphisms $f^* : H^i(Y, A) \to H^i(X, A)$.

In several situations an advantage of using cohomology over homology is that $H^*(X, A) := \bigoplus_i H^i(X, A)$ has a natural structure of an associative ring which makes
$f^* : H^*(Y, A) \to H^*(X, A)$ into a graded ring homomorphism. For example
$H^*(\mathbb{CP}^n, \mathbb{Q}) \cong \mathbb{Q}[t]/t^{n+1}$ with t in degree 2 of the grading.

The Lefschetz fixed point theorem can then also be phrased in the language of cohomology in exactly the same way as for homology. Using the additional ring structure, show that every continuous mapping $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point if n is even.

Exercise 6. *Aut(X) for a compact Riemann Surface of genus bigger than 1.*

A Riemann Surface X is a connected complex manifold of dimension 1. That is every neighbourhood of X is homeomorphic to an open subset of $\mathbb{C}$ using which one can make sense the notion of holomorphic functions between Riemann Surfaces. Using the classification of compact surfaces it is an easy fact that the underlying topological space of a compact Riemann Surface X is homeomorphic to $(T^2)^{\#g}$ for some $g \geq 0$. We call g the genus of the compact Riemann Surface X and denote it by $g(X)$.

Let $Aut(X)$ be the group of biholomorhic automorphisms of a compact Riemann Surface X. It is a fact that if $f \in Aut(X)$ then f induces the identity map on $H_2(X)$ (In other words f is orientation preserving). If $g(X) > 1$, using the Lefschetz-Hopf fixed point theorem, show that the induced action of $Aut(X)$ on $H_1(X)$ is faithful, ie. If $f_* : H_1(X) \to H_1(X)$ is the identity map then $f : X \to X$ is the identity on X.